Chapter 1: The Reals

Cummings' Exercises

Exercises with a $\star$ are considered "Notable Exercises" in the book.

Exercise 3.3 $\star$

This problem is to help you get a feel for the quantifiers in the definition of convergence (Definition 3.7). Your task: For each of the following definitions of *Nonverges*, give an example of a sequence (a_n) and a value a for which

- (a_n) does not converge to a (based on the real definition),
- (a_n) does *Nonverge* to a based on the definition given below.

Give a different example for each problem. For each of them, explain why your example works in a few sentences (no need to prove it completely). Your example for *Nonverges-type-4* should not work for *Nonverges-type-3*.

- **Definition 1:** The sequence (a_n) *Nonverges-type-1* to a if for all $\varepsilon > 0$ there exists some $n \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$.
- **Definition 2:** The sequence (a_n) *Nonverges-type-2* to a if for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that, for some $n > N$, we have $|a_n - a| < \varepsilon$.
- **Definition 3:** The sequence (a_n) *Nonverges-type-3* to a if there exists some $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there exists some $n > N$ such that $|a_n - a| < \varepsilon$.
- **Definition 4:** The sequence (a_n) *Nonverges-type-4* to a if there exists some $\varepsilon > 0$ and there exists some $N \in \mathbb{N}$ such that for some $n > N$ we have $|a_n - a| < \varepsilon$.

Exercise 3.5 $\star$

Give an example of a sequence (a_n) where a_n is negative for all n, and yet $a_n \to 0$.

Exercise 3.7 $\star$

Each of the following is an independent question, but for each suppose that the sequence (a_n) has the property that $a_n \in \mathbb{Z}$ for all n.

(a) Is it possible that $a_n \to 3.5$?

(b) If $a_n \neq a_m$ for all $n \neq m$, prove that (a_n) does not converge.

(c) If (a_n) converges, what can be said about this sequence?

Exercise 3.6 ⋆

(a) Consider the sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{1}{7}, \ldots$$

For which numbers L does the above have a subsequence converging to L?

(b) Does there exist a sequence (a_n) where, for every $L \in \mathbb{R}$, there is exists a subsequence of (a_n) which converges to L?

Exercise 3.19 ⋆

For each item, provide an example (and prove that it works) or prove that no such example exists.

(a) A sequence (a_n) where $6 < a_n < 7$ for all n, and which has a subsequence converging to 6 and also one converging to 7.

(b) A sequence (a_n) such that, for each $k \in \mathbb{N}$, there is a subsequence of (a_n) converging to $\frac{1}{k}$.

(c) A sequence (a_n) such that, for each $k \in \mathbb{N}$, there is a subsequence of (a_n) converging to $\frac{1}{k}$, but there is no subsequence of (a_n) converging to 0.

(d) A sequence (a_n) such that for every real number x, the sequence (a_n) has a subsequence that converges to x.

Exercise 3.31 ⋆

Exercise 3.14 ⋆

Let $a_0 = 2\sqrt{3}$ and $b_0 = 3$, and define two sequences recursively by

$$a_n = \frac{2a_{n-1} \cdot b_{n-1}}{a_{n-1} + b_{n-1}} \quad \text{and} \quad b_n = \sqrt{a_n \cdot b_{n-1}}.$$

Prove that (a_n) is monotonically decreasing and is convergent, and prove that (b_n) is monotonically increasing and convergent. Then prove that they converge to π.

Exercise 3.8 ⋆

Assume that (a_n) converges to some $a \in \mathbb{R}$ and (b_n) converges to some $b \in \mathbb{R}$. Also assume $c \in \mathbb{R}$.

(a) Prove that $(a_n + b_n)$ converges to $a + b$.

(b) Prove that $(c \cdot a_n)$ converges to $c \cdot a$.

Exercise 3.13 ⋆

Assume that (a_n) converges to a and (b_n) converges to b. Also assume $c \in \mathbb{R}$.

(a) Prove that $(a_n - b_n)$ converges to $a - b$.

(b) Prove that $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$, provided $b \neq 0$ and each $b_n \neq 0$.

Exercise 3.28 ⋆

Exercise 3.45 ⋆